E. Low Temperature Physics of 3D Ideal Fermi Gas

Context: Metals at $\underbrace{\text{room temperature}}_{kT_{room} \sim 0.024\,eV \sim 2.4\times 10^{-2}\,eV}$ ($T=0$ Physics gives $E_F \sim eV$)

$$kT_{room} \ll E_F \quad \underline{\text{OR}} \quad T_{room} \sim 300K \ll T_F = \frac{E_F}{k} \sim 10^4 K$$

$$T_{room}/T_F \sim 0.01 \sim 1\% \quad \text{(low-temperature physics)}$$

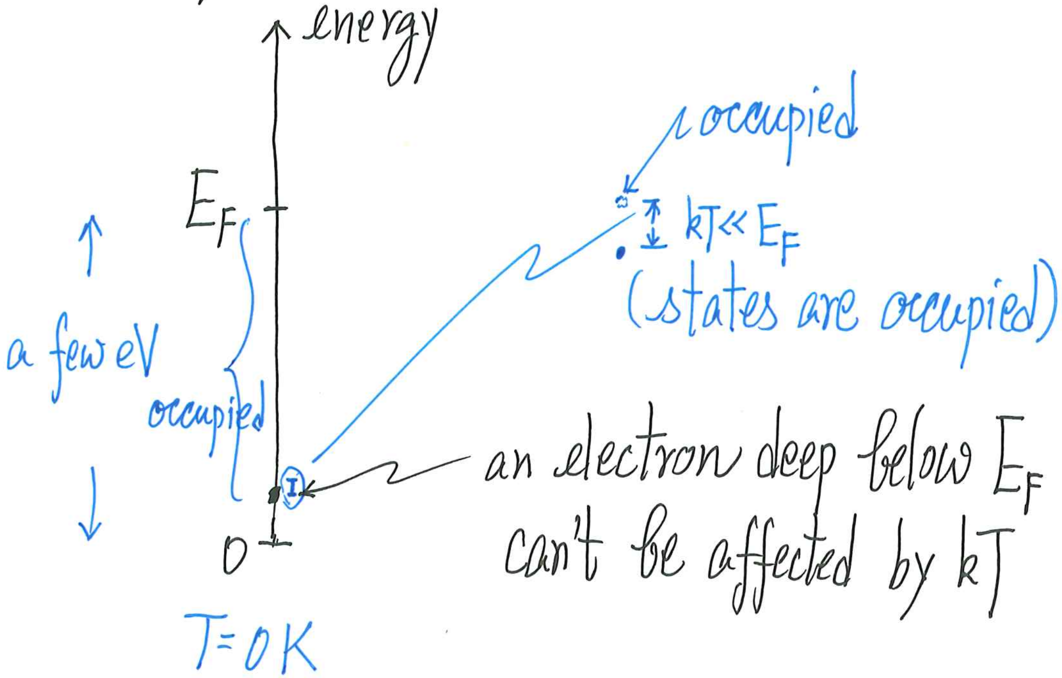
Q: "What is the physics of ideal Fermi Gas at $T \ll T_F$?"

Lazy Answer: It is almost that of $T=0$ physics!

It is 99% (or even better) correct!

But there is new physics to learn and physical phenomena to explain.

Physical Sense

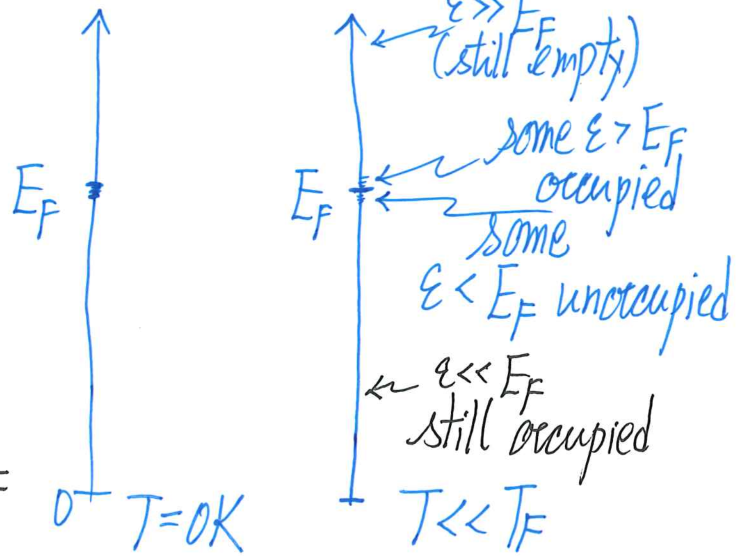


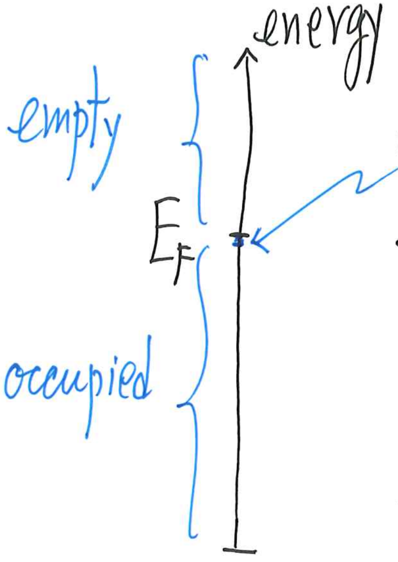
an electron deep below E_F can't be affected by kT

$kT \ll E_F$

$\Rightarrow$ most electrons cannot "take kT" to do something because of Pauli Exclusion rule

physics embedded in $f_{FD}(\varepsilon)$



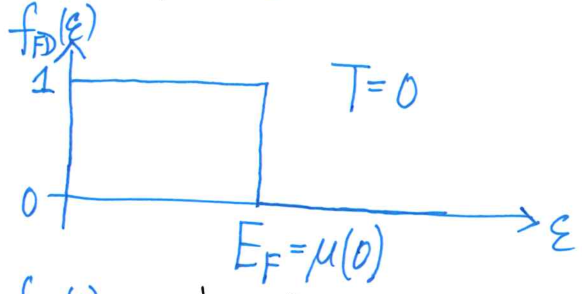


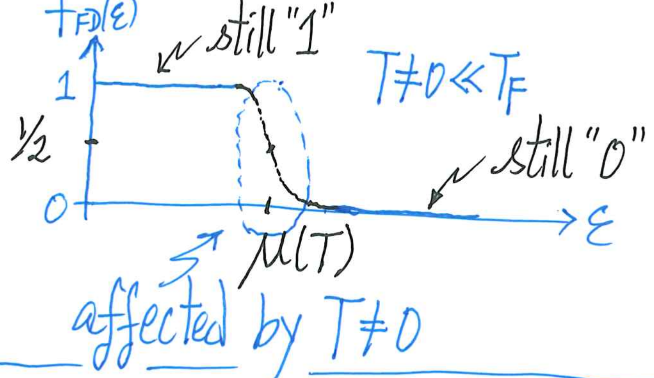
Only electrons occupying states of energy about kT around E_F will be affected, as they can go to the empty states about kT above E_F

$kT \ll E_F$ only Fermions at vicinity of E_F are affected (12)

This should have been embedded in $f_{FD}(\varepsilon)$

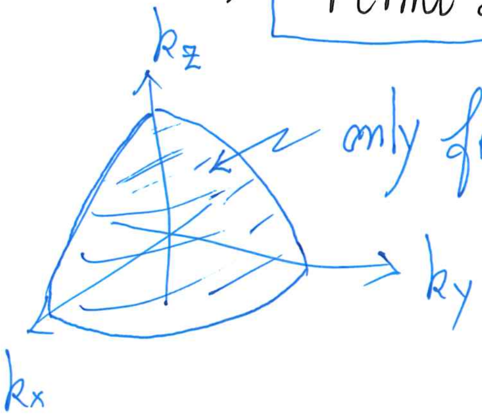
Only s.p. states in vicinity of E_F are affected (13)





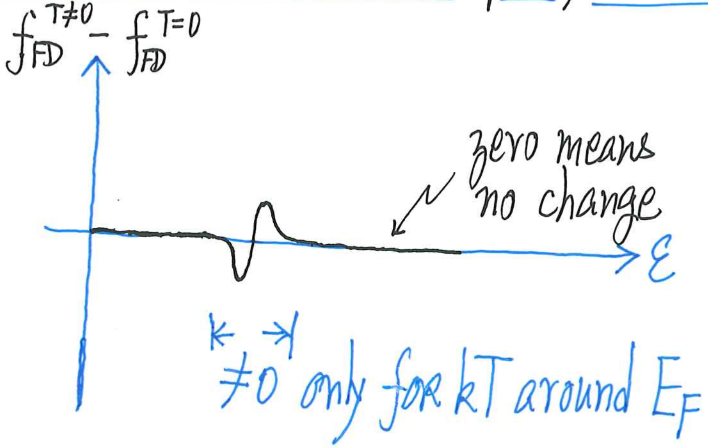
∴ $T\neq 0$ effects come from the fermions and states about E_F

⇒ "Fermi Surface" effect



only fermions "living" on or near Fermi Surface can be affected

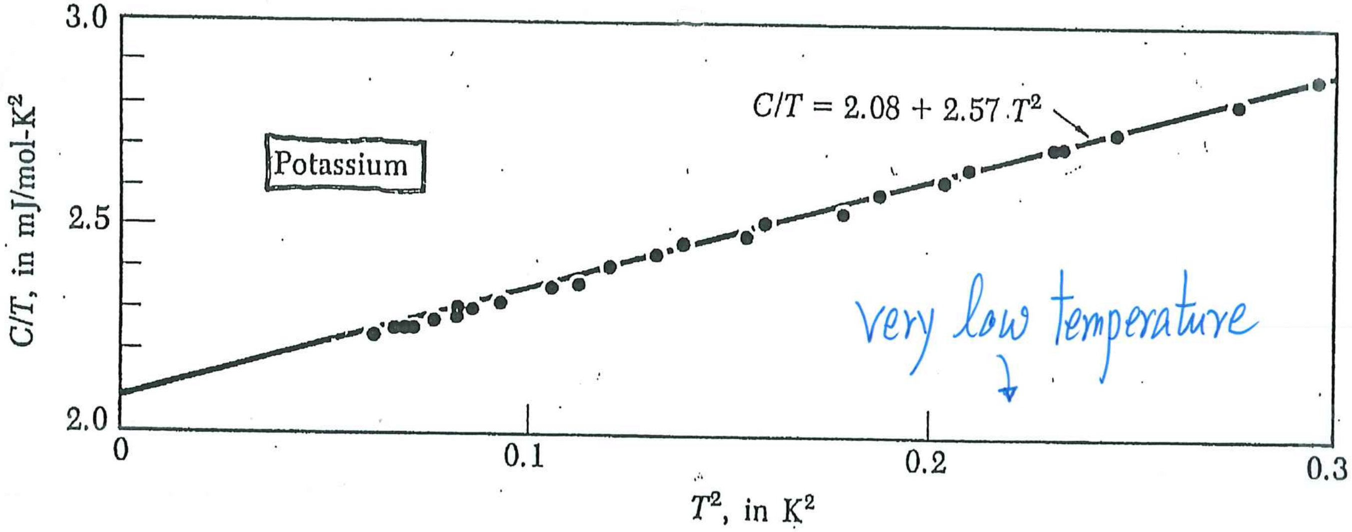
Key Question: where to put $\mu(T)$?



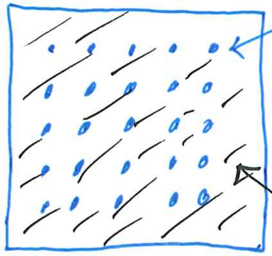
Physical Phenomena

C_V of metals: $$C = \underbrace{\gamma T}_{\text{electrons (fermions) contribution}} + \overbrace{bT^3}^{\text{vibrations of atoms in solid (Debye model)}}$$ (Why so?)

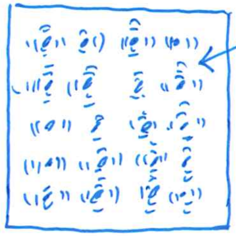
$$\frac{C}{T} = \gamma + bT^2 \Rightarrow \frac{C}{T} \text{ vs } T^2 \text{ is a straight line}$$



very low temperature



T≠0



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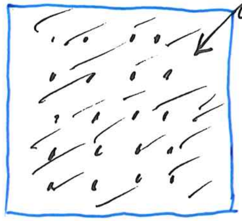


Strategy (No new strategy)

$$N = \int_0^\infty g(\varepsilon) \frac{1}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon \quad \text{locates } \mu(T) \qquad (1)$$

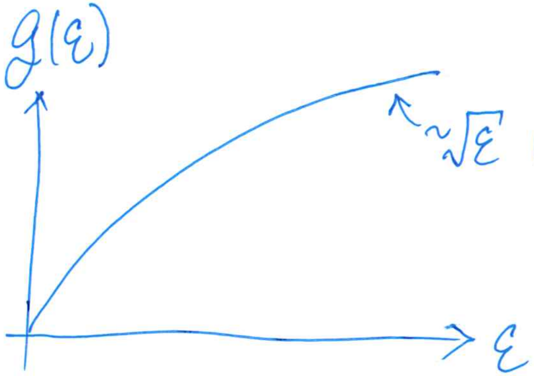
$$E = \int_0^\infty g(\varepsilon) \frac{\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon \quad \text{gives } E(T) \quad [\text{after knowing } \mu(T)] \qquad (2)$$

$$pV = \frac{2}{3} E \qquad (4)$$

Done!

<u>Recall</u>: $g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$

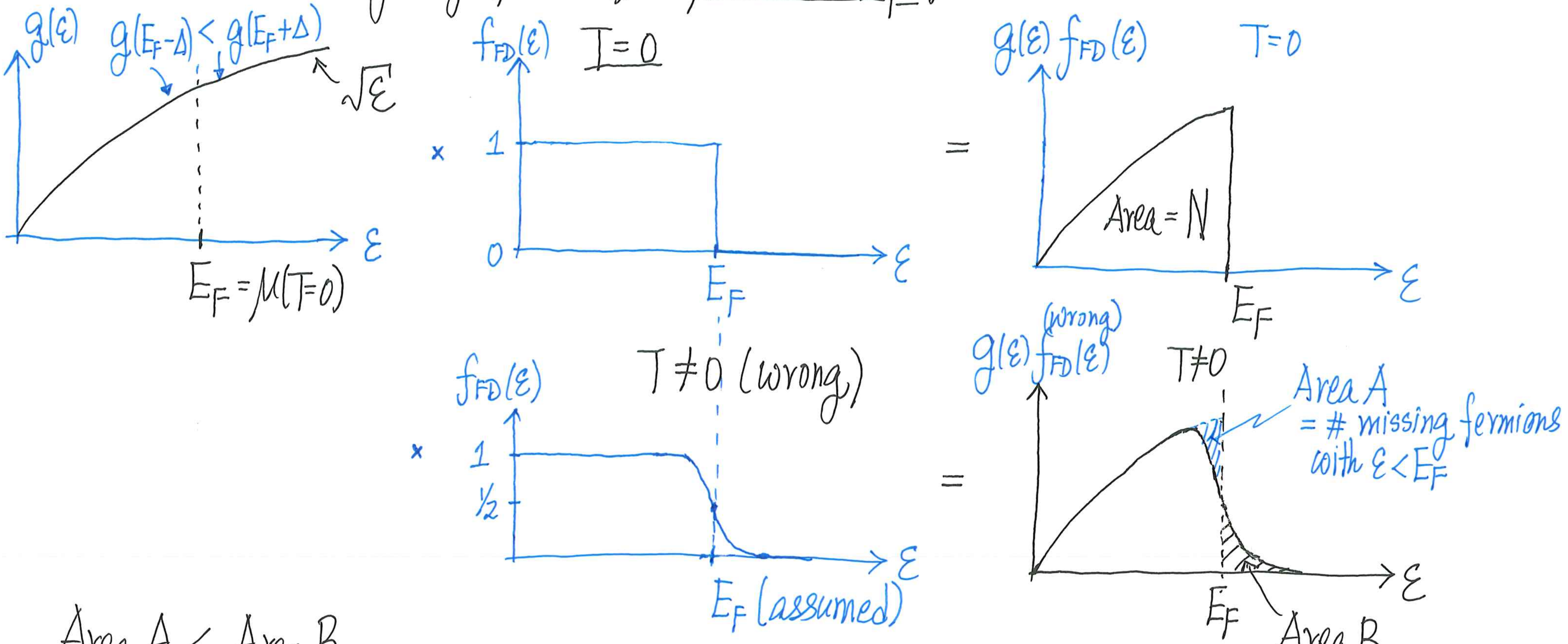
does NOT depend on temperature



$\sim\sqrt{\varepsilon}$ (increasing with ε as $\sqrt{\varepsilon}$)

(a) Key Concept : μ must shift with temperature, though NOT BY MUCH

What's wrong if $\mu(T\neq 0) = \mu(T=0) = E_F$?



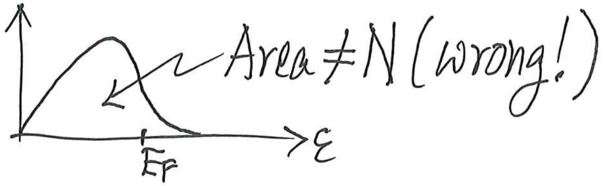
Area A < Area B

∵ $g(\varepsilon)$ is increasing with ε

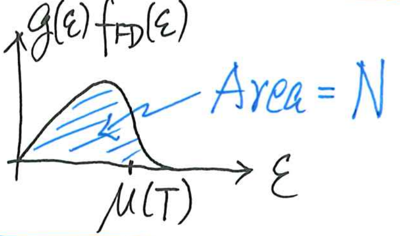
∴ $\boxed{\mu \text{ must shift with temperature}}$

But Area A < Area B

⇒



μ must be put at $\mu(T)$ so that



(14)

Shift μ to left (lower) or right (above) of E_F?

"Shift $\mu(T\neq 0)$ to $\mu(T\neq 0) > E_F$?" Even Worse!

∴ Should shift $\mu(T)$ to below E_F at finite temperature. (3D Fermi Gas)

[Ex.: How about 2D? 1D? Must think over them case-by case.]

Q: Shift $\mu(T)$ by how much?

Answer: Just a little bit!

$$\underbrace{\boxed{\mu(T) \cong E_F\left(1 \overset{\text{shifts down}}{-} \frac{\pi^2}{12}\overbrace{\left(\frac{kT}{E_F}\right)^2}^{(1\%)^2\sim 0.0001}\right)}}_{\text{Correction term is order }\left(\frac{kT}{E_F}\right)^2=\left(\frac{T}{T_F}\right)^2\text{ vs }1} \qquad (15)$$

Key idea mathematically

$$N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon$$ gives μ for given T

$\frac{N}{V}$ is known for a system

Same system at $T=0$ gives E_F (which gives an estimate on μ)

⋮ after many steps[+] $\left\{\begin{array}{l} kT \ll \mu \quad (\text{same as } kT \ll E_F) \\ \text{using a formula called } \underline{\text{Sommerfeld Expansion}} \end{array}\right.$

we get

$$\mu(T) \approx E_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right) \qquad (15)$$

thus the shifting down of μ from E_F is tiny.

[+] See Appendix A

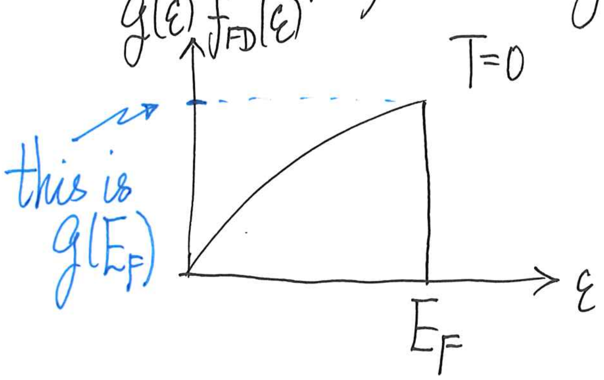
(b) Shift in Energy at $T \neq 0$ $(T \ll T_F)$

Recall: $E(T=0) = \int_0^{E_F} A\varepsilon^{3/2}\,d\varepsilon = \frac{2}{5} A E_F^{5/2} = \frac{3}{5} N E_F = \frac{2}{5} \underbrace{\overbrace{g(E_F)}^{AE_F^{1/2}}}_{\text{DOS at } \varepsilon = E_F} \cdot E_F^2$

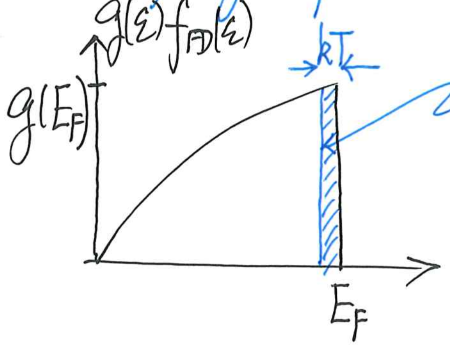
$T \neq 0$, $E(T)$ increases $\begin{cases} \text{by how much?} \\ \text{how does } \underbrace{E(T) \text{ depend on } T}_{} \text{?} \end{cases}$

To give $C_V = \gamma T$ (due to electrons) in experiments, $E(T) \sim T^2$ (why so?)

A physical argument on $E(T) = E(T=0) + (\text{some factor}) \cdot g(E_F) \cdot (kT)^2$



$g(E_F)$ = Density of (s.p.) states at the Fermi Energy

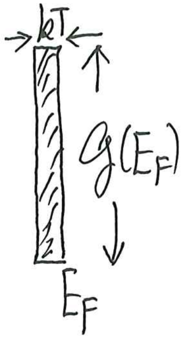


These are the fermions (kT around E_F) that can be affected by temp.

$T \neq 0$

(exaggerated, as $kT/E_F \sim 0.01$)

Number of fermions in 

≈ 

$= \underbrace{g(E_F)\cdot kT}_{\text{this is a number}}$

[Recall: $g(\varepsilon)\,d\varepsilon$ = # particles in $\varepsilon \to \varepsilon + d\varepsilon$]

Assuming each of these excited particles gains ~ kT in energy

$$\therefore \; E(T) \cong E(T=0) + \underbrace{\left[g(E_F)\cdot kT\right]\cdot kT}_{\text{increase in energy}^{+}\text{ (an estimate)}} \qquad (16)$$

$$\therefore \; C_V = \frac{dE}{dT} = 2\underset{\text{right unit}}{k}\,\Big[\underbrace{g(E_F)\cdot kT}_{\to \text{ a number}}\Big] \propto T \quad \Leftarrow \text{as observed experimently}$$

$$= \gamma T \qquad \Big(\text{with } \gamma = 2k^2 g(E_F) \propto \underbrace{g(E_F)}_{\text{a way to measure } g(E_F)}\Big) \qquad (17)$$

$^{+}$Increase in energy $g(E_F)(kT)^2$ is tiny compared with $E(T=0)$, which is $\sim g(E_F)\cdot E_F^2$, i.e. ratio is $(kT/E_F)^2 \sim 0.0001$

SOME ROUGH EXPERIMENTAL VALUES FOR THE COEFFICIENT OF THE LINEAR TERM IN T OF THE MOLAR SPECIFIC HEATS OF METALS, AND THE VALUES GIVEN BY SIMPLE FREE ELECTRON THEORY

ELEMENT	FREE ELECTRON γ (in 10^{-4} cal-mole^{-1}-K^{-2})	MEASURED γ (in 10^{-4} cal-mole^{-1}-K^{-2})	RATIO[a] (m^*/m)
Li	1.8	4.2	2.3
Na	2.6	3.5	1.3
K	4.0	4.7	1.2
Rb	4.6	5.8	1.3
Cs	5.3	7.7	1.5
Cu	1.2	1.6	1.3
Ag	1.5	1.6	1.1
Au	1.5	1.6	1.1
Be	1.2	0.5	0.42
Mg	2.4	3.2	1.3
Ca	3.6	6.5	1.8
Sr	4.3	8.7	2.0
Ba	4.7	6.5	1.4
Nb	1.6	20	12
Fe	1.5	12	8.0
Mn	1.5	40	27
Zn	1.8	1.4	0.78
Cd	2.3	1.7	0.74
Hg	2.4	5.0	2.1
Al	2.2	3.0	1.4
Ga	2.4	1.5	0.62
In	2.9	4.3	1.5
Tl	3.1	3.5	1.1
Sn	3.3	4.4	1.3
Pb	3.6	7.0	1.9
Bi	4.3	0.2	0.047
Sb	3.9	1.5	0.38

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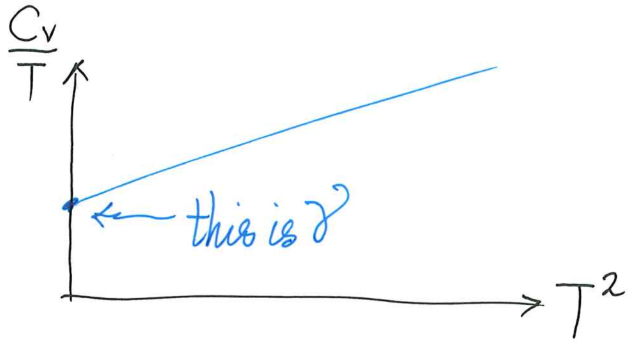
[a] Since the theoretical value of γ is proportional to the density of levels at the Fermi level, which in turn is proportional to the electronic mass m, one sometimes defines a specific heat effective mass m^* so that m^*/m is the ratio of the measured γ to the free electron γ. Beware of identifying this specific heat effective mass with any of the many other effective masses used in solid-state theory.



γ gives a measure on $\underbrace{g(E_F)}_{\text{an important Fermi surface parameter}}$

More formal calculation gives

$$E(T) = E(T=0) + \frac{\pi^2}{6}\, g(E_F)\cdot(kT)^2$$

and

$$C = \underbrace{\frac{\pi^2}{3}k^2 g(E_F)}_{\gamma_{\text{Fermi Gas}}}\cdot T = \gamma T$$

Key idea Mathematically

$$E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\infty}\frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu(T))/kT}+1}\,d\varepsilon$$ gives E for given T

($\mu(T)$ already fixed by "N-equation")

⋮ after many steps[†] (using Sommerfeld Expansion)

$$E(T) = E(T{=}0) + \frac{\pi^2}{6}\,g(E_F)\cdot(kT)^2 \qquad (18)$$

which gives a contribution to the heat capacity C as

$$C = \frac{\pi^2}{3}k^2\,g(E_F)\cdot T \equiv \gamma\,T$$ as observed experimentally (19)

and γ is a measure on $g(E_F)$ of a material

[†] See Appendix B

Remark: If we have mistaken the Fermi Gas as a Classical Gas,

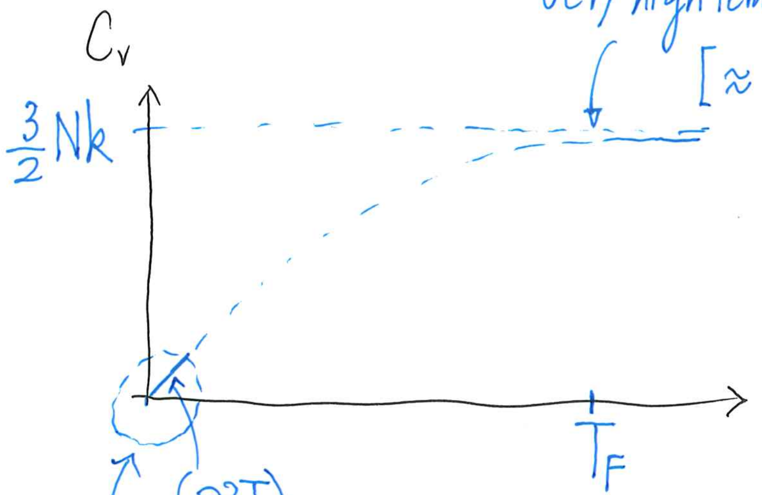
we would expect $E_{\text{classical}} = N \cdot 3 \cdot \frac{1}{2} kT$ (equipartition theorem)

$\therefore$ expect $C_{\text{classical}} = \frac{3}{2} Nk$ (independent of T)

$$\text{But } C_{\text{Fermi Gas}} = \frac{\pi^2}{3} g(E_F) \cdot k^2 T = \frac{\pi^2}{2} \cdot \underbrace{\left[\frac{2}{3} g(E_F) \cdot E_F\right]}_{N} \cdot k \left(\frac{kT}{E_F}\right)$$

$$= Nk \cdot \frac{\pi^2}{2} \cdot \left(\frac{kT}{E_F}\right)$$

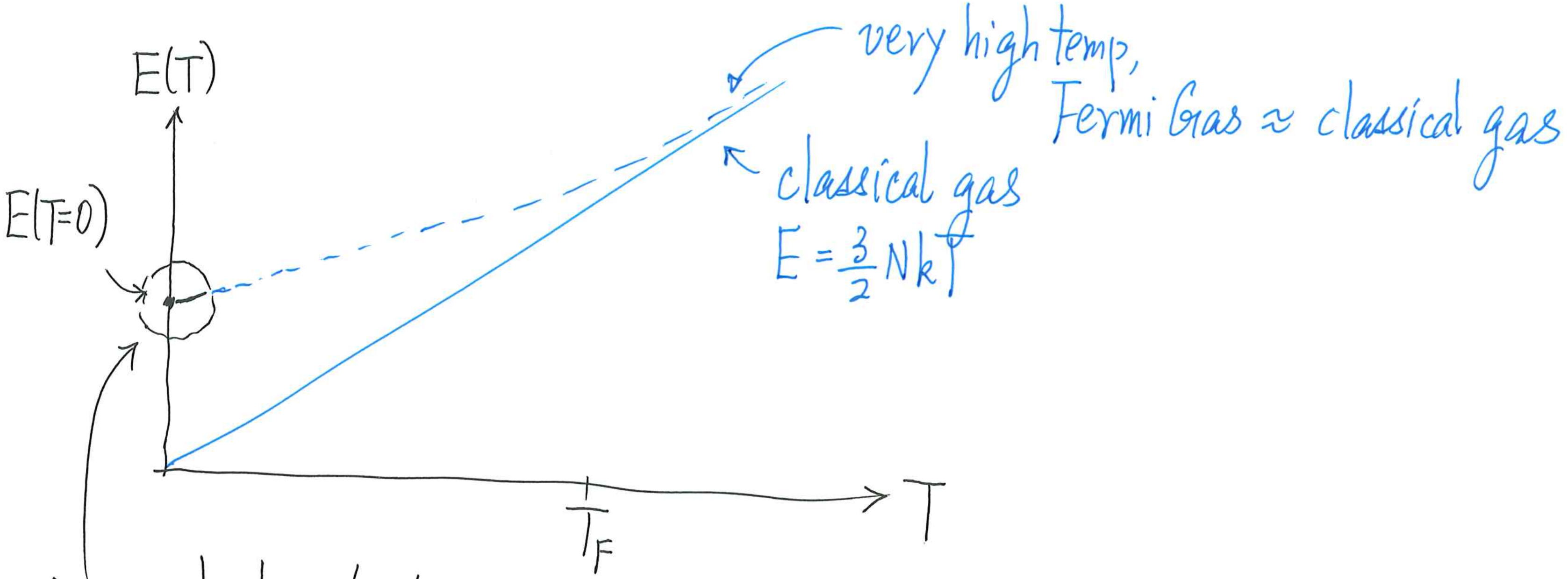
$$\approx \left(\frac{kT}{E_F}\right) \cdot C_{\text{classical}} \approx 1\% \cdot C_{\text{classical}}$$



very high temp $\sim T_F$ [$\approx$ classical gas]

(γT)

we worked out the physics here (Eq. (19))



We worked out the $T=0$ $(E(T=0))$ and $\underbrace{T \ll T_F\ (E(T))\ \text{physics}}_{\text{Eq. (18))}}$